Implementations of Probability Theory

Independent Study Report

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1 Objective

The educational focus of Implementations of Probability Theory surrounds the application of data models that produce non-deterministic insights through probabilistic methodology. By pursuing this study I hope to gain a deeper understanding of how to apply data in risk calculation for mitigation scenarios as they appear in real life, rather than the experimental lab conditions that enable algorithmic certainty.

In contrast to the path of black-box artificial intelligence and algorithms taught in **CSCI 335: Machine Learning**, this study is tailored to methods designed to produce confidence levels for uncertain events using certain terms, leveraging logical, traceable, and definite, calculations. Current course offerings in the realm of data science focus largely on the storing and management of data, and it is noted that the cluster of data science was until very recently under the branding of data management. Implementations of Probability Theory is intended to extend learnings in previous courses, notably **CSCI 420: Principles of Data Mining**, for more advanced algorithms used at the intersection of data and computing after the preprocessing stage.

After beginning this study the intended deliverable outline was determined to be technically implausible and has been replaced with demonstrations of applied algorithms. Taking inspiration from the retinal mosaic as displayed in **CSCI 431: Intro to Computer Vision** and discussion in **IGME 589: Computational Creativity and Algorithmic Art** on the appearance and nature of randomness in graphics, I will use this report as a platform for conceptual refactorization. These experiments are designed to appeal to human logical heuristics, helping them function as educational resources that develop a deeper understanding of why these systems work, not just the equations to use them.

2 Units

2.1 Unit 1: Statistics Review

To ensure a strong statistical foundation for the future learnings in probabilistic models, the first objective is to create a document outlining and defining key topics that are prerequisites for probabilities in statistics or for understanding generic analytical models. While not intended to be in-depth, the reported review can function as a topic recall and simplification dictionary.

2.1.1 Random Variables

1. **Discrete Random Variables -** values are selected by chance from a countable (including countably infinite) list of distinct values
2. **Continuous Random Variables -** values are selected by chance with an uncountable number of values within its range

2.1.2 Sample Space

A sample space is the set of all possible outcomes of an instance. For a six-sided dice roll event, the die may land with 1 through 6 dots facing upwards, hence:

$$S = [1, 2, 3, 4, 5, 6] \quad \text{where } S \text{ is the sample space}$$

2.1.3 Probability Axioms

There are three probability axioms:

1. **Non-negativity**:

   $$P(A) \geq 0 \quad \text{for any event } A,\ P(A) \in \mathbb{R}$$

   No event can be less likely to occur than an impossible event ($P(A) = 0$). P(A) is a real number. Paired with axiom 2 we can also conclude that $P(A) \leq 1$.

2. **Normalization**:

   $$P(S) = 1 \quad \text{where } S \text{ is the sample space}$$

   Unit Measure - All event probabilities in a sample space add up to 1. In essence, there is a 100% chance that one of the events in the sample space will occur.

3. **Additivity**:

$$P(A \cup B) = P(A) + P(B) \quad \text{if } A \cap B = \emptyset$$

A union between events that are mutually exclusive (events that cannot both happen for an instance) has a probability that is the sum of the associated event probabilities.

2.1.4 Expectations and Deviation

1. **Expectation -** The weighted average of the probabilities in the sample space

$$\sum^{S} P(A) * A = E \quad \text{where } E \text{ is the expected value}$$

2. **Variance -** The spread of possible values for a random variable, calculated as:

$$\sigma^2 = \frac{\sum(X - \mu)^2}{N}$$

Where N is the population size, μ is the population average, and X is each value in the population.
For samples, variance is calculated with **Bessel's Correction**, which increases the variance to avoid overfitting the sample:

$$s^2 = \frac{\sum(X - \bar{x})^2}{n - 1}$$

3. **Standard Deviation -** The square root of the variance, giving a measure of the average distance of each data point from the mean in the same units as the data.

$$\sigma = \sqrt{V} \quad \text{where variance is } V$$

2.1.5 Probability Functions

Probability Functions map the likelihood of random variables to be a specific value.

Probability Mass Functions

Probability Mass Functions (PMFs) map discrete random variables. For example, a six-sided die roll creates a uniform random PMF. Each side of the die has a one-sixth chance of landing face-up, so the discrete chances of each x value between 1 and 6 is

represented by a $\frac{1}{6}$th portion of the sample space:

$$P(A) = \begin{cases} 1/6 & \text{if} \quad X = 1 \\ 1/6 & \quad\quad X = 2 \\ 1/6 & \quad\quad X = 3 \\ 1/6 & \quad\quad X = 4 \\ 1/6 & \quad\quad X = 5 \\ 1/6 & \quad\quad X = 6 \end{cases}$$

Probability Density Functions

Probability Density Functions (PDFs) map continuous random variables. For example, this is a PDF representing a vehicle's risk of being stranded as it travels (in a line at a fixed speed). The y value increases as the vehicle puts distance between itself and the starting point but, once the halfway point is reached, the risk decreases as the distance between the vehicle and the destination decreases.

$$P(A) = \begin{cases} X & \text{if} \quad 0 \leq X \leq .5 \\ -X + 1 & \quad\quad .5 < X \leq 1 \\ 0 & \quad\quad otherwise \end{cases}$$

2.1.6 Limit Theorems

Law of Large Numbers

The Law of Large Numbers states that as the number of independent random samples increases, the average of the samples' means will approach the true mean of the population.

$$\text{true average} \approx \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \text{as } n \to \infty$$

Central Limit Theorem

The Central Limit Theorem states that the sampling distribution of a sample mean is a normal distribution even when the population distribution is not normal.

$$\frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\sigma} \xrightarrow{d} N(0,1)$$

Where X_i is the sample mean, $N(0,1)$ is a standard normal distribution, and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.
This is a challenging to understand solely as an equation. As an example, take a sample of two six-sided dice rolls and average their numbers. The more sample

averages taken, the more they will resemble a normal distribution where the majority of samples average around 3.5.

2.1.7 Confidence

Confidence is described using a confidence interval, which is a range of values that the true value is expected to be in, and its associated confidence level, which is a probability (expressed as a percentage) that the true value is in the confidence interval.

It is important to note that confidence levels, such as 95%, do not indicate that the real value is within 5% of the point estimate. The confidence level expresses the probability that the real value is in the range provided by the confidence interval.

At the highest level, calculating confidence intervals is simply the observed statistic (generally the mean) plus or minus the standard error. The percentage is identified by applying the z-score coefficient (in the case of nornal distribution, other distributions use non-parametric methods) that corresponds to that level of confidence. For instance, the z-multiplier for a confidence level of 95% is 1.96 so a confidence interval formula around the mean would look like this:

$$\text{interval} = \mu \pm (1.96 * \text{SE})$$

To calculate standard error when the population standard deviation (σ) is known:

$$\text{SE} = \frac{\sigma}{\sqrt{n}}$$

When σ is unknown:

$$\text{SE} = \frac{s}{\sqrt{n}}$$

where n is the size of the sample and s is the sample standard deviation. Notice how the standard error decreases with a larger sample size because it indicates a resilience in the sample to random events as per the Law of Large Numbers (2.1.6).

2.1.8 Statistical Inference

Statistical Inference is any data analysis to draw conclusions from a sample to make assertions about the population. Methods include estimation via averages and confidence intervals, and hypothesis testing, which attempts to invalidate (never *validate*) a hypothesis.

2.2 Unit 2: Probabilistic Theories and Epistemology

When developing probabilistic models it is vital to use domain expertise to expose the product to the full range of external variables that would be expected of a model applied to the real world. Without an appropriate understanding of both the limitations in research procedures and the true value of the data collected, the integrity of the model becomes inherently compromised.

As data scientists, we are uniquely at risk of falling for this trap because it is hard to fully grasp domain expertise when the nature of data science in a business setting frequently means consulting for many separate projects with a collectively massive scope. Of equal consideration, it is also easy to assume that the sophistication of our tools overrides imperfections in the data, in spite of mantras like 'Garbage In, Garbage Out'.

In this unit I explored some common fallacies and assumptions held by analysts who may not fully grasp the content that they work with, nor the problems they intend to solve. This required extensive research that I found was best digested in the form of books whose chapters chronicle multiple examples of a given principle. As such, the reading was not confined to just the timeslot designated for this unit. Research started during the months leading up to the start of the semester[1] and have continued through the independent study. This structure was particularly helpful to pull me back and gain perspective of what my goal was when I was knee-deep in feature construction and model formulation.

2.2.1 Moral Hazards and The Bob Rubin Trade

Picking pennies in front of a steamroller. When studying the effectiveness of a model the scope of review must capture the entire range of the sample space. Discarding black swans that don't impact the client does not mean the results will not reflect on the client for an oversight. There is therefore a question of obligation for data scientists to include flags for significant events in reality that do not effect the proposed course of action to the client.

The 2009 recession, attributed to the collapse of the housing market bubble, is the most common example of a moral hazard because the displacement of risk from banks who were federally required to give subprime loans to the taxpayer meant that banks could profit from subprime loans but would not be harmed when the inevitable occurred. In popular media, the housing bubble bursting is attributed to the banks where those in the industry passed off the event as something that nobody could have foreseen[2]. In reality, banks only ignored a probabilistic eventuality because

[1] Only research during the semester was logged in the timesheet

[2] For instance, in the 2015 movie *The Big Short*, only a few savvy traders who bothered to look into the details find that banks had, in their ignorance, built the bundled mortgages on an unstable foundation.

their models did not need to account for such an event.

Most emphasize the problems with risk transferrence when creating models. For this study's purposes, the important learning is that probabilistic models should not drop evaluations as soon as an event leaves the scope of the immediate client.

2.2.2 Ignoring Improbable Outliers with Outsized Impact

In machine learning it is common for algorithms to drop the most extreme (or a random selection of) datapoints to avoid overfitting and errors in data collection. One issue with the current implementation of this procedure is that it is often done blindly, ignorant of information that these outliers may relay. For instance, in a selection of 300 water samples from a stream, all but a few show a normal amount of oxygen in the stream. A citizen scientist may discount the remaining pockets as a statistical implausibility that is most likely indicative of a failure in sample testing and drop the most extreme 5% of datapoints. However, if these few pockets show a complete disruption of the dissolution process, the vast majority of aquatic life in the stream will eventually pass through these pockets without oxygen and die, resulting in an outsized impact from just a few sources.

Nassim Taleb in *Fooled By Randomness* describes this event with an analogy to Russian Roulette: If there was a 5/6 chance of winning a million dollars and a 1/6 chance of killing yourself, many people would at least hesitate before pulling the trigger. But what if the barrel is 10,000 rounds and it was only a 1/10,000 chance of harm? In this case, many less-than-rational actors use the game repeatedly to acquire wealth indefinitely, forgetting or even outright ignorant that eventually the unlikely, or, as the actor would see it, the unthinkable, happens and all of the gains are completely negated.

2.2.3 Fooled By Randomness

While most statisticians are familiar with techniques to remove noise to get a clearer picture of long-term trends, many forget that noise over longer terms can materialize as highly improbable events. For instance, it is improbable to flip a fair coin and have heads land face up 5 times in a row, but if the coin is flipped millions of times, it's exceedingly unlikely that a 5-head sequence does not occur.

In Nassim Taleb's namesake book, *Fooled By Randomness*, this concept is applied to ongoing timeseries analysis in stock markets. By accounting for the scope of the prior evidence, Taleb models the probability that daily events are the effect of noise, a number that remains high even in the face of multiple point swings in the market. Understanding this chance is critical because often observers attempt to justify random market events to events with high publicity that in reality had a negligible on the market, fooling investors out of acting on prices deviating from their

target.

2.2.4 Lindy Effect

The Lindy Effect describes the importance of historical evidence of continuity when estimating its continuity in the future. For items with a set lifespan, such as perishable goods, each passing day is indicative of a shorter remaining life expectancy, but the same is not true for nonperishables like tools and concepts. For example, consider the lifespan of a news story or hot book. Many such stories may take the world by storm, but then be nearly forgotten months later. However, older writings are incredibly unlikely to be forgotten in the next few months. It would be truly bizzare if everyone decided Shakespeare was not worth learning in the next few years because its value has been determined for so long to be high enough to maintain its popularity.

Applying this concept to probability theory, information and evidence that has been important for a long time is likely to stick around long after hot new examples or tactics that contradict it fade into obscurity. When measuring risk of startups, the concept and foundations may indeed be strong, but they have to be contrasted with the robustness of past ideas as proven over time. This concept also has applications for how people think about new things in their day to day life. In the news and papers outlining new developments, "Inaccurate science... is constantly being published. The Lindy-conscious consumer of scientific data will take seriously only information that has held up over a period of time"[3] because time has removed uncertainty associated with volatility of untested (or tested less than the alternative) information.

2.2.5 Decision Theory

Decision theory is the study of how people make decisions with uncertain information. There are two main branches of decision theory:

Normative/Rational Decision Theory

This branch studies how people *should* make decisions. In problems with other actors, as in game theory, it is assumed that all other actors will also act with perfect rationality, allowing for precise calculation of the actions of all of the others and their expected utility to the agent.

Descriptive Decision Theory

This branch studies how people actually make decisions which includes factors such as psychological and emotional biases. It applies subjective value measurements, frequently working in parallel with Dempster-Shafer Theory (2.2.7).

[3] `https://www.nytimes.com/2021/06/17/style/lindy.html`

2.2.6 Info Gap Decisions

In info gap decision theory there is not enough information to assign probabilities to events. The goal, then, is to select a course of action that is robust in the face of uncertainty. Where decision theory can predict expectations in irrationality to determine expected values, info gap decisions approximate the range of probabilities and weight them to estimate expected value. In essence, it applies probabilities to probabilities, adding an additional layer to insulate calculations from a lack of data or lack of understanding of a topic. Tying this into the Lindy Effect (2.2.4), we can compare the large range of probabilities of new, untested information with the narrower range from old, tested information which has experienced more challenges, just as confidence increases with a larger sample size.

2.2.7 Dempster-Shafer Theory

This section is an extra theory chosen to coincide with the unit 3 focus on Bayesian statistics. The Dempster-Shafer theory is a derivative application of Bayes Theorem (2.3.1) where subjective beliefs are applied to independent variables not tracked by the belief network. Shafer so eloquently describes this process by supposing that two friends, both of whom he subjectively believes are 90% reliable, tell him that a limb has fallen on his car [4]. Without observing Shafer's car we can calculate that there is only a 1% chance that both friends are unreliable, so there's a high liklihood that the statement is true.

However, if both friends are unreliable, they are not necessarily lying. Thus, there is actually less than 1% chance that a limb fell on the car. The exact probability can only be calculated by determining how likely it is that the friends would find it funny to tell Shafer that a limb fell on his car, contrasted with the odds that such a friend may also be willing to throw limbs at his car so as to maintain their ever-reliable facade. If one also considers the possibility that Shafer's friends mistakenly believed a limb fell on his car, this uncertainty must also be combined with the evidence for the most accurate picture.

2.2.8 Minority Rule through Renormalization

One way that details about a sample can be supressed is through minority rule, where analyses is skewed by the influence of a small subsection of the population imposing attributes onto a pliable, but larger subsection of the population. Often used in social sciences and asymmetric warfare, the stubbornness of a handful of people, say, those with a demanding preference for organic foods, requires the surrounding environment to adapt. Most people who do not eat organic but would not object if it was all that

[4] `http://glennshafer.com/assets/downloads/articles/article48.pdf`

was offered. Thus, a family with a single person with a dietary preference can flip the entire kitchen to fit that preference. This process is called renormalization and it runs counter to the observations of outsiders that might infer that the whole family prefers organic foods.

Scaled upwards, the renormalization effect might then apply itself to a cookout between families who acknowledge one family has a dietary preference. That might then renormalize the entire community, resulting in local grocery store offerings being near-exclusive to the dietary preference of a remarkably small portion of the community. If a data scientist then infers from the offerings of this grocery store the dietary preferences of the community, they would be inclined to believe that the actual minority is not just a majority, but a requirement amongst the population. In this sense, tolerance for intolerance begets intolerance.

2.2.9 Scale as a Dimension

Just as the rate and plausibility of renormalization is impacted by the ratio of the minority to the flexible majority, other interactions can become more complex through scale to the same effect as the curse of dimensionality. The curse of dimensionality is a reference to the exponential complexity of solving a problem with x variables. Two boolean variables, each containing one of two values, has 4 possible combinations of values. A third variable doubles this number to 8, a fourth doubles it again to 16. In complex interactions, scale acts as its own source of dimensionality becuase each new node in an ecosystem can interact with each pre-existing node, influencing interactions between it and another pre-existing node, which then influences the interactions from that node, and so forth.

In *Skin in the Game*, Taleb uses the example of neuroscience to show the improbability of AI ever reflecting the full complexity of the human brain. He acknowledges advancements in neuoscience that accurately models interactions between neurons in the human brain, but scaling ths up to replicate human behavior is not so easy. While binary variables apply an exponential effect with a base of 2 (2^x where x is number of binary variables), neurons interlock and may have an effect of a hundredth, thousandth, or even millionth base.

This complexity, Taleb says, explains why even carefully studied brains of worms with only 300 neurons are still too complex to really understand, let alone simulate. If neurons had only a binary effect, the complexity could be calculated to $2^300 = 2 * 10^90$ which, while massive, could conceivably be computed in the distant future. However, if each neuron can interact with just 5 others, the combination explosion grows to $(2^5)^300 = 3.5 * 10^451$. Applying Moore's Law and we assume that a society's computational capacity doubles every two years, it would take 2400 years before this

difference in computational power could be rectified:

$$2 * \log_2 \left(\frac{3.5 * 10^{451}}{2 * 10^{90}} \right) \approx 2400.04 \text{ years}$$

Not to mention, neuron interactions are incredibly complex, containing dimensions in of themselves, not binary values. Good luck computing that, robot overlords.

2.2.10 Methodology Considerations

As another homage to *Garbage In, Garbage Out*, I'd like to present some instances of methodology creating useless data for the target variables. This is not just a reference to bad studies, such as those that try to measure social behaviors, oblivious to the fact that participant observation alters their behavior. There are many instances that data can be untainted but used without appropriate context. In particular, *The Signal and the Noise* and *Fooled by Randomness* highlight many instances where timeseries studies believe that decades of historical data is necessicarily comprehensive. Financial events in particular are often labelled as unpredictable by experts only when their models fail because the context of a national economy changes dramatically which can reveal attributes to market economics that were previously obscured by practices that isolate those variables. An event never occurring in history does not discount its possiblity of occurring in the future. Similarly, events that may have been impossible in the past are not necessarily impossible in the future. As an extreme example to prove a point, consider the following:

> I have taken 10134023 instances of the last 40 years, during all of which Obama has been alive. Therefore, with so much time passed and many trials, I can say with a high degree of certainty that Obama is immortal.

Silly, yes, but it is easy to become detached from context points when you begin digging deep into mathematical models. Data science is generally considered to be the intersection of coding, statistics, and domain knowledge, implying domain knowledge is secondary to computational ability. I'd argue just the opposite - incomplete knowledge of contemporary models still lends itself to effective data analysis but an incomplete understanding of what is being measured is dangerous and potentially counterproductive.

2.3 Unit 3: Bayesian Statistics

This unit was deliberately separated from statistical review due to the percieved complexity of the topic and the magnitude of usage in recent data science breakthroughs. Bayes Theorem is a part of the cirriculum for both **MATH 351 - Probability and Statistics** and **CSCI 420 - Principles of Data Mining**. However, as both approached the topic from different perspectives and while neither solidified my personal confidence in its use, I chose to take extra time to learn this important topic in my own way.

It has been said that statistics does not come naturally to the human brain, hence statistics is, by mathematical standards, a young discipline. Resulting research on Bayesian statistics has led me to the conclusion that the opposite may be true - Bayes Theorem is quite intuitive, but its discipline has not had the time to crystallize best practices for instructing it. For instance, updating one's beliefs to compare probabilities with the number of documented occurrences is frequently used in philosophical discussion in the form of explanations that subsets with high liklihood of fufilling terms are valid classifications even when the subset size results in overall fufilled terms to be infrequently categorized as the proposed subset. Most people understand these expressions but, when shown a table and how to calculate those ratios, the content enters the realm of collegiate instruction.

2.3.1 Bayes Theorem

Bayes Theorem is a rule for conditional probability that calculates the probability of a cause given an event has occurred. The equation for Bayes Theorem is as follows:

$$P(A|E) = \frac{P(A) * P(E|A)}{P(A) * P(E|A) + (1 - P(A)) * P(E|\neg A)}$$

This formula appears more complex as it is. The denominator, while directly translating to "The probability of A times the probability of event E occuring given A divided by the probability of A times the probability of event E occuring in A plus the probability of not A times the probability of E occuring in not A" can be more easily expressed as $P(E)$ or the probability of event E occuring:

$$P(A|E) = \frac{P(A) * P(E|A)}{P(E)}$$

Finally, this equation is updated to replace descriptions with technical terms:

$$\text{Posterior Probability} = \frac{\text{prior} * \text{likelihood}}{\text{Evidence}}$$

By utilizing venacular more familiar to everyday life, Bayes Theorem can be translated as:

$$\text{P(occurence stems from A)} = \frac{\text{\# of occurences from A}}{\text{total \# of occurences}}$$

To appeal to mental visualization, the sample space can be imagined geometrically as a 1 unit by 1 unit square[5]. The area of this square, 1 unit squared, represents a probability of 1 (or 100%) and the probability of any possible outcome fits inside this square. Intuitively, this visualization can also be thought of as a confusion matrix where the squares are drawn proportional to their representative probabilities.

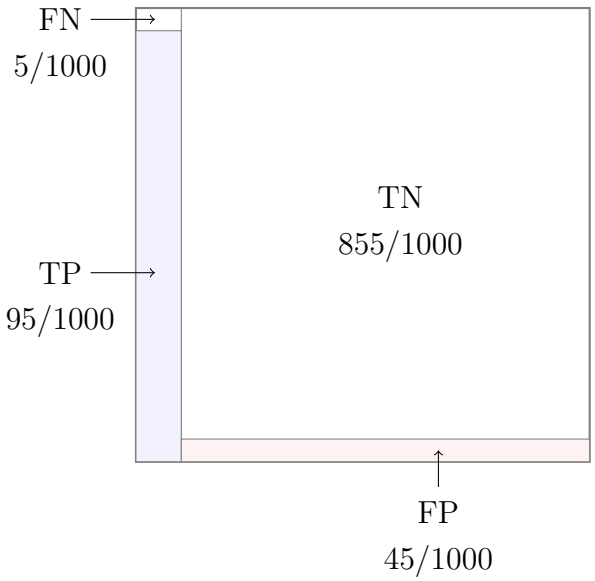
Consider an example where a patient wants to know if their positive cancer test is actually a false negative. Reviewing the test history, it's found to be accurate 95% across 1,000 uses. Given that we want to find the chances that a positive test is truly from a patient with cancer, let's highlight only the cases where a test is positive. A confusion matrix for this example would look like this:

	Cancer (100 patients)	No Cancer (900 patients)
Negative	False Negatives 5 patients	True Negatives 855 patients
Positive	True Positives 95 patients	False Positives 45 patients

Notice that the test does make the correct identification 95% of the time (and in this example, 95% regardless of actual value) but that there are almost half as many false positives as there are true positives, meaning having a positive test is not representative of a 95% chance of having cancer.

Proportinally scaling the probability matrix squares to create the sample space square defined earlier, we can see that the TP box appears to be approximately twice the size of the FP box. Logically, then, if we chose a random positive test, there's a two-thirds chance of the patient selected being from the true positive category:

[5]Concept credit to 3Blue1Brown on Youtube, this video is what finally clarified in my mind what the frankly simple equation behind Bayes Theorem meant.
`https://www.youtube.com/watch?v=HZGCoVF3YvM`



Bayes Theorem as applied to this problem can be simply expressed as:

$$P(\text{has cancer given positive test}) = \frac{\text{TP}}{\text{TP} + \text{FP}} = \frac{\frac{95}{1000}}{\frac{95}{1000} + \frac{45}{1000}} = 67.9\%$$

Meaning that, given a random positive test, there is a 67.9% chance of the patient actually having cancer, not far off from the two-thirds visual trick.

2.3.2 Bayesian Updating

Bayesian Updating is another term that has been added to buzzword vocabulary to describe a process that isn't directly related to Bayesian Statistics but appears to have been rediscovered by academia through study of applied Bayes Theorem. In essence, Bayesian Updating simply states that observed occurrences should not override previous evidence and that it should instead be added to it in equal weight (equal value being a naive assumption). This evidence updating makes applications of Bayes Theory calculate posterior probabilities continuously as new information enters the system rather than a frequentist approach where the calculation only performed once.

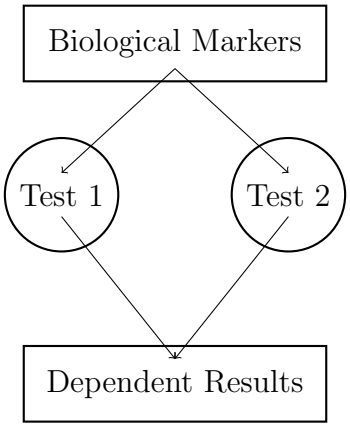
2.3.3 Bayesian Belief Networks

Using Bayes to build an ensemble of models

Bayesian Belief Networks are probabilistic graphical models that preserve conditional dependence between random variables. In spite of its name, Bayesian Belief Networks do not necessarily apply Bayesian models, though they are a way to utilize Bayes Theorem for domains with greater complexity beyond a single posterior probability.

In this type of network, edges are directed and the structure is utilized in a single direction. This is in contrast to undirected Hidden Markov Models (to be covered in the next unit) that do not assume the order of aquisition of random variables. While it may not be practical to calculate the full conditional probability of a variable, Bayesian Belief Networks allow us to identify conditionally dependent variables that are weighted on the basis of an earlier random variable.

Following the example in the Bayes Theorem section of this report (2.3.1), let's suppose that a patient with a positive test takes a hypothetical second test. However, the second test's results are partially dependent on the first since they measure overlapping biological markers.



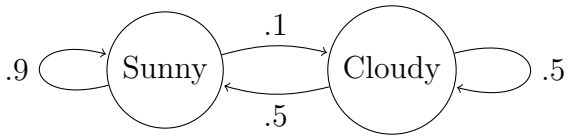
Test 1 Result	Test 2 Result	P(A)
Prior beliefs of test 1		
Unknown	Unknown	10%
Positive	Unknown	67.857%
Negative	Unknown	0.581%
Prior beliefs of test 2		
Unknown	Positive	55%
Unknown	Negative	1%
Dependent results from both tests		
Positive	Positive	75%
Positive	Negative	1.5%
Negative	Positive	0.6%
Negative	Negative	0.087%

Note that this probability of positive results in both tests (which both have greater than 50% of positives being true positives) is only equally certain as two positives from two independent tests each with 50% of positives being true. If the dependence was not included in the calculation and we ignored the fact that the tests partially measure the same thing, as would have occured in a Naive Bayes model, the tests' combined accuracy would be unjustly inflated.

2.4 Unit 4: Markov Methods

2.4.1 Markov Chains

Markov Chains are a form of probabilistic automaton where the likelihood of transitioning to a new state depends solely on the current state with no memory of prior states. For example[6], suppose a weather prediction program wants to know whether tomorrow will be a sunny or cloudy day, based on the current weather. Using the current weather as a state, the program identifies that there is a 10% chance of a sunny day transitioning into a cloudy day and a 50% chance that a cloudy day transitions into a sunny day:



Note that there is no information preserved between steps. Markov Chains are memoryless, so any information that must be available to them must be expressed as the state, such as the sunny and cloudy states in the example above. Accemically, this is called the **Markov Assumption**, though it is vocabulary that can easily be explained with few additional words and won't be used for the rest of this paper. One benefit of such a straightforward structure is that it enables easy calculation of the probabilities of reaching a state k-steps from the current position. By expressing the chain as a transition matrix where rows represent the current state, the column represents the next state, and each cell contains the probability of the state moving from the column state to the row state, we get a 1-step transition matrix:

$$\begin{pmatrix} .9 & .1 \\ .5 & .5 \end{pmatrix}$$

or, expressed as a table:

Current State	Next: Sunny	Next: Cloudy
Sunny	90%	10%
Cloudy	50%	50%

To turn this into a k-steps transition matrix, this 1-step matrix only needs to be raised to the k-th power:

$$\begin{pmatrix} .9 & .1 \\ .5 & .5 \end{pmatrix}^k$$

[6]example sourced from:
`https://towardsdatascience.com/introduction-to-markov-chains-50da3645a50d`

To find the probability of the weather two days from the current state, plug 2 into k:

$$\begin{pmatrix} .9 & .1 \\ .5 & .5 \end{pmatrix}^2 = \begin{pmatrix} .86 & .14 \\ .7 & .3 \end{pmatrix}$$

From this matrix we can determine that if it is currently sunny, there is a 86% chance that it will be sunny in two days and, if it is currently cloudy, there is a 70% chance that it will be sunny in two days. As k approaches infinity, the model approaches its equilibrium where the starting state becomes irrelevant. In this example, any random day would be 83.333% likely to be sunny, representative of the long-term behavior of the system (climate), so the matrix of the equilibrium looks like this:

$$\begin{pmatrix} .9 & .1 \\ .5 & .5 \end{pmatrix}^\infty \approx \begin{pmatrix} .83333 & .16666 \\ .83333 & .16666 \end{pmatrix} \text{ OR: } \begin{pmatrix} .83333 \\ .16666 \end{pmatrix}$$

2.4.2 Hidden Markov Models

In contrast to the visible Markov Models above, Hidden Markov Models cannot observe the states within the model. The benefit to using such a model is that observations of occurrences can use alogirthms such as the Viterbi Algorithm to determine the probability of a sequence of observations and estimate which state is active in a given instance. These results extrapolating process to the result is reminiscent of inverse problems and many explanatory uses of data science, such as in finance where, with the benefit of hindsight, analysts work to determine why events unfolded the way they did.

In addition to states, initial state probabilities, and transition probabilities, Hidden Markov Models also utilize observations, and emission probabilities, or the probability of an observation given a transition from state a to b. Using the earlier example where states represent either a sunny or cloudy day, an observation liklihood matrix can be created for a weather sensor that can only determine if the ground is wet. On a cloudy day there is a probability of rain and thus a high probability of the ground being wet, whereas a sunny day would not nearly as often be triggered by dew or sensor tampering:

$$\begin{array}{r} \\ \text{Sunny} \\ \text{Cloudy} \end{array} \begin{array}{c} \begin{array}{cc} \text{dry} & \text{wet} \end{array} \\ \begin{bmatrix} .95 & .05 \\ .6 & .4 \end{bmatrix} \end{array}$$

Thus, an observation sequence may look like this:

[Dry, Dry, Wet]

In this case, it can be confidently assumed that the wet signal is representative of a rainy, cloudy day. In contrast, we can only be moderately confident that the two dry days leading up to it were sunny days. Intuitively, it is most likely that there were two sunny days followed by a rainy day. By multiplying the probability of observation to the transformation to the potential state, the probability of occurrence is revealed. For the purposes of the example we will use the 83%-16% equilibrium matrix from earlier as the initialization matrix to reflect the random chance of any given day being sunny or cloudy:

Three consecutive sunny days:

$$(\frac{5}{6} * .95) * (.9 * .95) * (.9 * .05) \approx 0.03$$

Three consecutive cloudy days:

$$(\frac{1}{6} * .6) * (.5 * .6) * (.5 * .4) = 0.006$$

Sunny, sunny, cloudy:

$$(\frac{5}{6} * .95) * (.9 * .95) * (.1 * .4) \approx 0.027$$

Interestingly, the calculation reveals that it is actually more probable that there was an unusual wet third day during a sunny streak than for there to have been a cloudy day following two sunny days.[7]

Brief sidenote, since the probability initial state is not known, the probability of initalization at state n is expressed in calculations as π_n. I will not use this notation in this report because I think it is confusing and somewhat ridiculous to have mathematical notation with as ubiquitous and universally constant a meaning as π be addressed for something that has no relation to the constant. Whatever convention made this determination is seriously damaging the accessibility of mathematics for anybody shy of a walking computational index.

2.4.3 Viterbi Algorithm

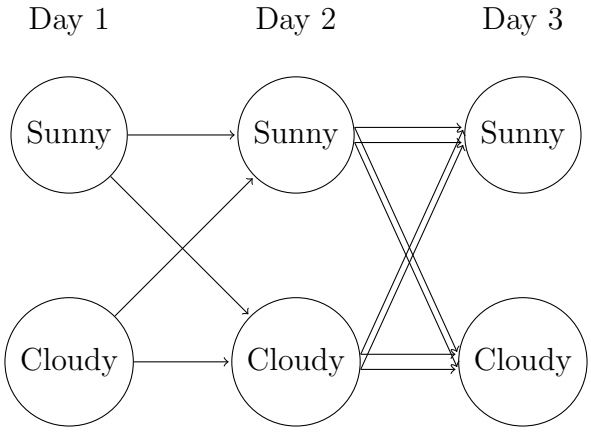
Markov is memoryless - only the most probable sequence to a state matters

While it is feasible to calculate the probabilities for each possible route to a series of observations, such a process produces an exponential time complexity. With each state change, the number of paths to keep track of grows exponentially, which in practical terms means countless threads on each state separated only by the history

[7] I say interesting because I forgot how low I set the probability of sunny to cloudy and wholly expected the intuitive sun-sun-cloud answer to prove accurate. Math moment.

of how they got there. Enter the Viterbi Algorithm, which reduces the effect of a step (or, as in our example, a new day) from an exponential relationship ($O(N^T)$) to a flat multiple ($O(N^2T)$). This is possible because the Viterbi Algorithm creates partial solutions by eliminating all but the most optimal branch to reach the next state instead of recomputing each exit from a state for each entry. If a route is deemed improbable, it will not be considered the next time the same observation sequence occurs at that state.

More intuitively, consider that there are multiple ways to reach a given state in 1 step. Once each path's probability is computed, you only need to retain the highest probability path to that state and the next step will only require calculation from that state once.[8] Consider the following graphic rendition of each possible 3-day sequence of sunny vs cloudy:



Notice that there are two arrows from each day 2 state to each day 3 state because there two paths were created to reach each of the day 2 states. If there was a fourth day depicted, there would be 4 calculations from each day 3 state to each day 4 state. To prevent this, the Viterbi Algorithm only preserves the most likely path to each node. For instance, there are two paths to a sunny day on day 2. Either the first day was sunny and it stayed sunny, or the first day was cloudy but transitioned to sunny the next day. Using the same [Dry, Dry, Wet] observation sequence as before, the probabilities of these paths occurring can be calculated:

Two consecutive sunny days:

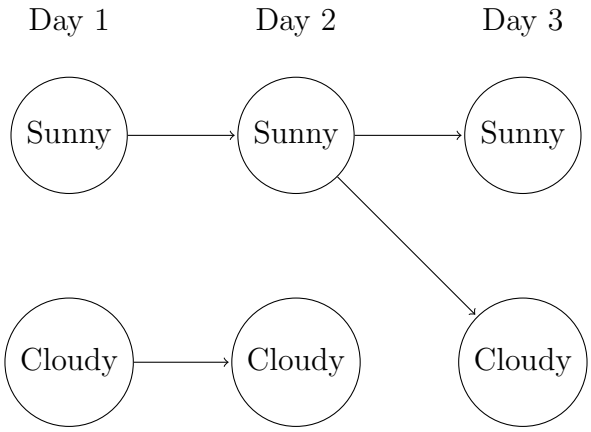
$$(\frac{5}{6} * .95) * (.9 * .95) \approx 0.677$$

Sunny, cloudy:

$$(\frac{1}{6} * .6) * (.1 * .6) = 0.006$$

[8]The mathematical notation to describe this algorithm is criminally challenging to parse. I want to acknowledge this video for being the only one of its kind that did not rely on the notation: `https://www.youtube.com/watch?v=6JVqutwtzmo`

Hence, we can eliminate the [Cloudy, Sunny] starting sequence from the most probable sequence of steps given the observations. Doing the same thing for the rest of the visualization leaves fewer arrows and therefore fewer calculations:



With only two sequences remaining, the final comparison needs only to determine if it is more likely for there to have been three consecutive sunny days or a sequence of two sunny days and a cloudy day[9], which we already calculated in the Hidden Markov Model section (2.4.2). If this calculation was extended to include additional days, the Viterbi Algorithm would never need to calculate a path that started with two cloudy days because all branches stemming from that route have already been pruned by the third day.

[9]Had we assumed a 50-50 chance of initialization on a sunny or cloudy day, the probability of three consecutive cloudy days would have been more likely than a sunny, sunny, cloudy sequence. Yet another example where contextual completeness in the methodology makes a significant improvement in accuracy over what might otherwise have been napkin math.

2.5 Unit 5: Monte Carlo Simulations

Monte Carlo Simulations are models that directly recreate the conditions of an environment containing random variables to simulate the outcome given a value in place of the random variable. This placeholder value may be an average of an expected occurrence but often the simulation is run many times with a randomly selected value so the results can be analyzed in place of many trials in the real environment.

Monte Carlo is useful when interactions between many variables produce deterministic but intractable results or if the steps to translate into a deterministic model are not fully understood. For every probability problem there exists a Monte Carlo Simulation that steps through the process of how a result is created without any derived formulation (which may be incorrect, especially if a problem is not completely understood). While the results are influenced by short-term bias in the random variable, the results converge towards the true Probability Mass Function (2.1.5) as long as the simulation accurately reflects the interaction between variables.

2.5.1 How To Make a Monte Carlo Simulation

If you've ever created a simulation and run it multiple times to get a feel for what is most likely to happen, congratulations! You've created a Monte Carlo Simulation.

As an example, consider the scenario described in the Markov Model section of this report (2.4) where we want to predict if a day x days in the future will be either sunny or rainy. Here is that same table representing the odds of a day transitioning from the state of the previous day:

Current State	Next: Sunny	Next: Cloudy
Sunny	90%	10%
Cloudy	50%	50%

To run a single possiblity of this interaction, initialize the state to define if the first day is sunny or cloudy (possibly using the equilibrium matrix as discussed prevously). Then, generate a random number and partition the possible results to match the table. If the first day is sunny and the random number is between 0 and 1 then one option is to transition to a cloudy state if the number is greater than .9, reflecting the 90% chance that the next day will also be sunny. Continuing this for the next few days, the random variable may leave a state transition path like [Sunny, Sunny, Cloudy]. Running the simulation again may net a different path: [Sunny, Cloudy, Sunny]. With more simulations, the collected random sample will quantify the probability of a sunny day on the third day with a simple ratio:

$$\frac{\#\text{ of simulations that end with a sunny day}}{\text{total }\#\text{ of simulations}} \approx 0.86\%^{10}$$

[10] Again, assuming a 100% chance of sunny day initialization.

We can validate this model by using our k-step transition matrix (2.4):

$$\begin{pmatrix} .9 & .1 \\ .5 & .5 \end{pmatrix}^2 = \begin{pmatrix} .86 & .14 \\ .7 & .3 \end{pmatrix}$$

Recall the top left number of this matrix reflects the probability of ending on a sunny day (column) given that the first day was sunny (row).

2.5.2 Monte Carlo Integration

Monte Carlo Integration is one use of Monte Carlo Simulations where the area of an object (or graphical integral) is calculated by selecting random coordinates and calculating the ratio of random coordinate points that were in the object (under the curve) to the total number of random coordinates. I'm including this section in the report for completeness since when I drafted this study's schedule I incorrectly assumed that this was a topic that would extend Monte Carlo, not just apply it.[11]

One example of this integration method, called Buffon's Needle, is an approximation of pi (yes, π) by dropping sticks on a series of parallel lines. Assuming the length of the sticks is shorter than the distance between the parallel lines, this interaction is statistically governed by the expression $\frac{2l}{\pi d}$ where l is the length of the sticks and d is the space between parallel lines[12].

2.5.3 Markov Chain Monte Carlo (MCMC) methods

Simulations can depend on their prior results

MCMCs are a class of Monte Carlo simulations that epitomize stochastic sampling. Given a probability distribution that is too complex to be analyzed traditionally, MCMCs approximate the target distribution with an equilibrim distribution that converges on that target distribution.

Contrary to the name of "Markov Chain Monte Carlos" and most educational works on the topic, I believe the easiest way to understand MCMC as a Monte Carlo simulation with 1-step memory. MCMC invokes the name of Markov Chains because in the array of sampled random values each value is randomly selected with influence of the previous value - something many compare to the memoryless state-hopping in Markov Chains. In reality, the 'state' in MCMCs is just a value whose importance is in how often this value is in the array. It's not a state with contextual value or an associated transition matrix.

[11] I made this mistake at least twice. If you're bored, try to spot which topics they are. Unlicensed gamification moment.

[12] Learn more about and run a Monte Carlo Simulation of the sticks approximation at `https://prancer.physics.louisville.edu/modules/pi/index.html`

There are a number of algorithms that implement the concept of MCMC, the most common of which is called the **Metropolis-Hastings Algorithm**. [13] In this variation, an initial value is selected at random. For each step, another random value, frequently in the range of one standard distribution, is added to this number, which has a $\frac{P(\text{new value})}{P(\text{new value})+P(\text{current value})}$ chance of becoming the new current value and added to the list of samples. If the current sample is selected over the new value, the current sample is added a second time to the list of samples. This acceptance criteria directs the samples towards high probability events while still keeping open the chance of the samples bridging the gap between local probabilistic maxima.

[13] If you're like me and can't handle the abstractions that education by mathematical notation requires, this video on Metropolis-Hastings is the best I can point you to on the topic of MCMCs: `https://www.youtube.com/watch?v=oX2wIGSn4jY`

3 Retrospective Discussion

At the end of this independent study it's worth reflecting on how my initial proposal has changed as I've learned more about this topic. Going into the Fall 2024 semester I wanted to understand how complex algorithms manage the influence of untracked variables and how they could be used to derive formulas for the influence on tracked variables on the target. While I did recieve some insight on how to go about formulating experiments to do this, especially through a more personal understanding with the foundational statistics, I found fairly little industry application of the scientific conceptualization that I expected. Most practical applications of probability theory rely less on an in-depth understanding of a scenario's component interactions and more on building a model that is robust to what it does not understand. Instead of removing noise, probablistic techniques work within the noise and are capable of correcting when noise leads it to make an incorrect assessment.

I still believe in the value of probability to track underlying and derivative features. In the future I will be considering the development of multivariate and noise isolation techniques. In executing this study when I did, not only will the content I learned will be fresh in my mind for when I start my Data Science graduate classes next month, but the unresolved curiosities that it uncovered will also be given a chance to develop. I'm already half-expecting one of the projects that I thought up for this to end up in my thesis. If in 2 years I publish some model derived on intelligent action of random but structured agents, you'll know that something this semester stuck.

A major challenge of this study was sifting through the mountains for of educational resources that rely on obscure mathematical notation with monumental complexity. It is simply an unfathomable failure on behalf of the educational systems that instruct probability to convey intuitive algorithms with an archaic language that nobody speaks. It felt like striking gold when I finally found the one resource that graphically or even programmatically translates these formulas. Most of my research time was dedicated to interpreting educational resources that appeared to have been made to appease superior instructors rather than making an effort to instruct. There may have been hours spent in research of confidence intervals, Bayes Theorem, and the Viterbi Algorithm, but there was ultimately only a single article or video for each of these topics that bridged the gap between abstraction to conceptualization.

I want to propagate this treasure and wrote this report to utilize those methods of instruction - not through mathematical abstractions of memory but through description. I am very proud of my newfound skills writing expressions and creating graphics in LaTeX but even here I disjointed and rejoined each calculation with textual explanation, just as one would comment code in any remotely complex function. Mathematicians should not be exempt from this procedure. Additionally, I structured my report to be comprehensive, down to the order of axiom review. Content relevant

to a section is either found in previous sections or simply described such that there isn't even a need for the actual academic terminology. While there is little expectation that this report will be read by anyone seeking to learn these concepts, I very much hope to hone the explanatory qualities that I have started here and share them with future students.

There may not have been a major application project as we'd originally intended for this independent study but I feel what came out of it has made my understanding of probability theory more grounded in how it's actually used than if I had made some niche demonstration that was poorly thought out in its viability. I'd like to thank my advisor, Dr. Kinsman, for seeing this endeavor for what it is and by encouraging me to keep up the research in its natural direction. This flexibility and uncertain guidance is exactly what is needed from data scientists if we are to truly find the unseen gems in our experiments. With the indefinite optimism that is lacking the world over, take confidence in the solutions not yet found.

4 Appendix Information

Given that this report may only be shared by the RIT Computer Science Department without the appendix, the appendix for this report, to include the timesheet and tasks completed for this independent study, will be made available as a separate document.